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ASYMPTOTIC DIMENSION, DECOMPOSITION COMPLEXITY, AND HAVER'S PROPERTY C

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ABSTRACT. The notion of the decomposition complexity was introduced in [GTY] using a game theoretical approach. We introduce a notion of straight decomposition complexity and compare it with the original as well with the asymptotic property C. Then we define a game theoretical analog of Haver's property C in the classical dimension theory and compare it with the original.

1. INTRODUCTION

Asymptotic dimension was introduced by Gromov to study finitely generated groups though the definition can be applied for all metric spaces [Gr1]. It received a great deal of attention when Goulyang Yu proved the Novikov Higher signature conjecture for groups with finite asymptotic dimension [Yu1]. There were many other similar results about groups and manifolds under assumption of finiteness of the asymptotic dimension or the asymptotic dimension of the fundamental group [Dr2],[Ba],[CG],[DFW]. When G. Yu introduced property A and proved the coarse Baum-Connes for groups with property A, it was a natural problem to check whether every finitely presented group has this property. A construction of a finitely presented group without property A was suggested by Gromov [Gr2] (see for detailed presentation [AD]). Gromov's random group construction is an existence theorem. Still, it is a good question whether a given group (or class of groups) has property A. The answer is unknown for the Thompson group F.

It turns out that the dimension theoretical approach to verification of property A proved to be quite productive. There are different extensions of features of finite asymptotic dimension to asymptotically infinite dimensional spaces. Some of them came from the analogy with classical dimension theory, some from function growth, and some

from game theory. The asymptotic property C was defined in [Dr1] by analogy with Haver's property C in the classical dimension theory. It was shown that the asymptotic property C implies the property A. Different versions of asymptotic dimension growth were suggested in [Dr3],[CFY],[DS1],[DS2]. The property A was proven for groups with the sublinear dimension growth [Dr1],[CFY], then for the polynomial dimension growth [Dr3], and finally, for the subexponential dimension growth [Oz]. Since dimension growth of a finitely generated group is at most exponential, this leaves a question whether the property A is equivalent to a subexponential dimension growth.

The notion of decomposition complexity of a metric space was introduced in [GTY] (see also [NY]) in game theoretic terms. It was shown that the finite decomposition complexity (FDC) implies property A. The finite decomposition complexity was verified for a large class of groups, in particular for all countable subgroups of $GL(n, K)$ for an arbitrary field K [GTY]. In this paper we address the question what is the relation between asymptotic property C and FDC. It turns out that in the classical dimension theory there is no analog of FDC. We plan to present one and compare it with Haver's property C in a future publication. Here to make a comparison of FDC and the asymptotic property C we introduce the notion of the straight finite decomposition complexity sFDC opposed to the game theoretic finite decomposition complexity=gFDC=FDC. We prove the following implications

$$gFDC \ \Rightarrow \ sFDC \ \Rightarrow \ property \ A$$

and

$$asymptotic \ property \ C \ \Rightarrow sFDC.$$

We know that the last implication is not an equivalence for metric spaces. We expect that is not an equivalence for groups as well (see Question 4.3). We do not know if any of the first two implications is revertible.

In Section 5 of the paper we compare a game-theoretic approach with the standard in the classical dimension theory. We did the comparison for the Haver's property C and found that a game-theoretic analog of it defines the countable dimensionality. It is known that these classes are different [E].

2. Preliminaries

All spaces are assumed to be metrizable.

A generic metric is denoted by d. Given two nonempty subsets A, B of X, we let $d(A, B) = \inf\{d(a, b) \mid a \in A, \ b \in B\}$.

Let $R > 0$. We say that a family $\mathcal{A}$ of nonempty subsets of X is R-disjoint if $d(A, B) > R$, for every $A, B \in \mathcal{A}$.

A metric space X is geodesic if for every $x, y \in X$ there exists an isometric embedding $\alpha \colon [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$.

A metric space X is discrete if there exists $C > 0$ such that $d(x, y) \geq C$, for every $x, y \in X$, $x \neq y$. A discrete metric space is said to be of bounded geometry if there exists a function $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that every ball of radius r contains at most $f(r)$ points.

Let $\mathcal{X}, \mathcal{Y}$ be families of metric spaces and $R > 0$. We say that $\mathcal{X}$ is R-decomposable over $\mathcal{Y}$ if, for any $X \in \mathcal{X}$, $X = \bigcup(\mathcal{V}_1 \cup \mathcal{V}_2)$, where $\mathcal{V}_1, \mathcal{V}_2$ are R-disjoint families and $\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{Y}$.

A family $\mathcal{X}$ of metric spaces is said to be *bounded* if

$$\text{mesh}(\mathcal{X}) = \sup\{\text{diam}\, X \mid X \in \mathcal{X}\} < \infty.$$

Let $\mathfrak{A}$ be a collection of metric families. A metric family $\mathcal{X}$ is *decomposable over* $\mathfrak{A}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{A}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$.

2.1. **Definition.** [GTY] We introduce the metric decomposition game of two players, a defender and a challenger. Let $\mathcal{X} = \mathcal{Y}_0$ be the starting family. On the first turn the challenger asserts $R_1 > 0$, the defender responds by exhibiting an R_1-decomposition of $\mathcal{Y}_0$ over a new metric family $\mathcal{Y}_1$. On the second turn, the challenger asserts an integer r_2, the defender responds by exhibiting an R_2-decomposition of $\mathcal{Y}_1$ over a new metric family $\mathcal{Y}_2$. The game continues in this way, turn after turn, and ends if and when the defender produces a bounded family. In this case the defender has won.

A metric family $\mathcal{X}$ has FDC if the defender has always the winning strategy. A metric space X has FDC if so is the family $\{X\}$.

2.2. **Definition.** We say that a metric space X satisfies the *straight Finite Decomposition Property* (sFDC) if, for any sequence $R_1 < R_2 < \ldots$ of positive numbers, there exists $n \in \mathbb{N}$ and metric families such that $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ such that $\{X\}$ is R_1-decomposable over $\mathcal{V}_1$, $\mathcal{V}_i$ is R_i-decomposable over $\mathcal{V}_{i+1}$, $i = 1, \ldots, n-1$, and the family $\mathcal{V}_n$ is bounded.

The following easy follows from the definition.

2.3. **Proposition.** *Suppose that X has the FDC. Then it has straight FDC.*

We recall that the *asymptotic dimension* of a metric space does not exceed n, asdim $X \leq n$ if for every $R > 0$ there are uniformly bounded

R-disjoint families $\mathcal{U}_i$, $i = 0, \ldots, n$ of sets in X such that the family $\cup_{i=1}^n \mathcal{U}_i$ is a cover of X [Gr1].

The following notion was introduced in [Dr1].

2.4. Definition. A metric space X is said to have the *asymptotic property C* if for every sequence $R_1 < R_2 < \ldots$ there exists $n \in \mathbb{N}$ and uniformly bounded R_i-disjoint families $\mathcal{U}_i$, $i = 1, \ldots, n$, such that the family $\cup_{i=1}^n \mathcal{U}_i$ is a cover of X.

We recall the definiton of the property A [Yu2].

2.5. Definition. Let X be a discrete metric space. We say that X has Property A if for every $\varepsilon > 0$ and $R > 0$ there exists a collection of finite subsets $\{A_x\}_{x \in X}$, $A_x \subset X \times \mathbb{N}$ and a constant $S > 0$ such that

(1) $\frac{\#(A_x \Delta A_y)}{\#(A_x \setminus A_y)} \leq \varepsilon$ when $d(x, y) < R$.
(2) $A_x \subset B(x, S) \times \mathbb{N}$.

Let X be a discrete metric space with bounded geometry. It is known that X has Property A if and only if for every $\varepsilon > 0$ and every $R > 0$ there exists a map $x \mapsto \xi_x \colon X \to \ell_{1,+}(X)$ and a number $S > 0$ such that

(1) $\|\xi_x - \xi_y\|_1 < \varepsilon$ if $d(x, y) < R$ and
(2) $\mathrm{supp}(\xi_x) \subset B(x, S)$

(see, e.g., [HR]).

A map $f \colon X \to Y$ of metric spaces is bornologous (coarse uniform) if there exist a function $\varphi \colon [0, \infty) \to [0, \infty)$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$. A map is metrically proper if the preimage of every bounded set is bounded. A map is coarse if it is both coarse uniform and metrically proper.

We say that two maps $f, g \colon X \to Y$ are close if

$$d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\} < \infty.$$

A map $f \colon X \to Y$ is a coarse equivalence if there exists a map $g \colon Y \to X$ such that the compositions gf and fg are close to the identity maps 1_X and 1_Y respectively.

These notions of coarse geometry have their counterparts for the families of metric spaces. A map $F \colon \mathcal{X} \to \mathcal{Y}$ of a metric family $\mathcal{X}$ into a metric family $\mathcal{Y}$ is a collection of maps $F = \{f\}$, where every f maps some $X \in \mathcal{X}$ into some $Y \in \mathcal{Y}$.

We say that $F \colon \mathcal{X} \to \mathcal{Y}$ is coarse uniform if there exists a nondecreasing function $\varrho \colon [0, \infty) \to [0, \infty)$ such that $d(f(x), f(y)) \leq \varrho(d(x, y))$, for every $f \in F$ and every x, y in the domain of f.

A family $F\colon \mathcal{X} \to \mathcal{Y}$ is metrically proper if there exists a nondecreasing function $\delta\colon [0,\infty) \to [0,\infty)$ such that $\delta(f(x), f(y)) \leq d(x,y)$, for every $f \in F$ and every x, y in the domain of f.

3. Straight Finite decomposition complexity

3.1. **Theorem** (Coarse Invariance). *The property sFDC is coarse invariant.*

Proof. Let $f\colon X \to Y$ be a coarse equivalence and let Y have the sFDC. Let $R_1 < R_2 < \ldots$. By [GTY, Lemma 3.1.1], there exists $S_1 > 0$ such that, whenever $Y = (\cup \mathcal{Y}_1) \sqcup (\cup \mathcal{Y}_2)$ is an S_1-decomposition of Y, then $X = (\cup \mathcal{X}_1) \sqcup (\cup \mathcal{X}_2)$, where $\mathcal{X}_i = \{f^{-1}(Z) \mid Y \in \mathcal{Y}_i\}$, $i = 1, 2$ is an R_1-decomposition of Y.

Next, we apply [GTY, Lemma 3.1.1] (which is actually formulated for metric families) to the maps $F_i\colon \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$, and we obtain that there exists $S_2 > 0$ and an S_2-decomposition of every $Z \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ such that the preimages of the elements of this decomposition form an R_2-decomposition of $f^{-1}(Z) \in \mathcal{X}_1 \cup \mathcal{X}_2$. We proceed similarly until the elements of the S_2-decomposition of every $Z \in \mathcal{Y}_{n-1} \cup \mathcal{Y}_{n-1}$ form a bounded metric family. Then their preimages also form a bounded metric family, by [GTY, Lemma 3.1.2]. $\qquad\square$

3.2. **Proposition.** *Asymptotic property C implies straight FDC.*

Proof. Given a sequence $R_1 < R_2 < R_3 < \ldots$, apply property C condition to get $\mathcal{U}_1, \ldots, \mathcal{U}_m$, R_i-disjoint uniformly bounded families that cover X. Then we define the partition of X into $\mathcal{U}_1$ and the complement (a single tile partition). Then consider the intersection of $\mathcal{U}_2$ with the complement, and so on. $\qquad\square$

3.3. **Proposition.** *Every discrete metric sFDC space can be isometrically embedded into a geodesic metric sFDC space.*

Proof. We follow the construction from [Dr1]. Let X be a discrete metric sFDC space. For any $x, y \in X$, attach isometrically the metric segment $[0, d(x,y)]$ to X along its endpoints. Endow the obtained space, X', by the length metric. We assume that X naturally lies in X'.

Let $R_1 < R_2 < \ldots$ be a sequence of positive numbers. Let

$$K_n = \{x \in X' \mid n(R_1 + 1) \leq d(x, X) < (n+1)(R_1 + 1)\}.$$

Note that every K_n is coarsely equivalent to X. Indeed, K_n lies in the $(n+1)(R_1 + 1)$-neighborhood of X. On the other hand, since

X is unbounded, for every $x \in X$ there is $y \in K_n$ with $d(x, y) < (n+1)(R_1+1)$, i.e. X lies in the $(n+1)(R_1+1)$-neighborhood of K_n. Therefore, X and K_n are of finite Hausdorff distance and therefore coarsely equivalent.

We represent X' as $\bigcup(\mathcal{V}_1 \cup \mathcal{V}_2)$, where

$$\mathcal{V}_1 = \{X\} \cup \{K_n \mid n = 2i + 1, \ i \in \mathbb{N}\},$$

$$\mathcal{V}_2 = \{K_n \mid n = 2i, \ i \in \{0\} \cup \mathbb{N}\}.$$

Now the result follows from the fact that X has sFDC and Proposition 3.1. $\qquad\square$

3.4. **Theorem.** *Straight FDC implies property A for metric spaces.*

Proof. Given $n \in \mathbb{N}$, we apply the definition of FDC to X with the sequence $R_i = 4^i n$ to obtain a nested sequence of partitions $\mathcal{F}_1 > \mathcal{F}_2 > \cdots > \mathcal{F}_m$ that ends with a uniformly bounded partition. Recall that the partition $\mathcal{F}_1$ splits in two R_1-disjoint families, and, generally, each tile $F \in \mathcal{F}_i$ is decomposed into two R_{i+1}-disjoint families. We construct a sequence of open covers $\mathcal{V}_i$, $i = 1, \ldots, m$ of X by enlarging the tiles of the partitions $\mathcal{F}_i$. For tiles F in $\mathcal{F}_1$ we take the open R_1-neighborhoods $V_F = N_{R_1}(F)$. For each tile $F \in \mathcal{F}_2$ we take $V_F = N_{R_2}(F) \cap N_{R_1}(F')$ where $F' \in \mathcal{F}_1$ is the unique tile such that F is defined by the partitioning of F' and so on. Thus for each $V \in \mathcal{F}_{i+1}$ there is a uniquely determined $V' \in \mathcal{V}_i$ such that $V \subset V'$. We will denote this relation as $V' = p(V)$. Also we adopt the notation $p^k(U) = p \ldots p(U)$ for the k times iteration with the convention $p^{m+1}(U) = X$ and $p^0(U) = U$.

For each $V \in \mathcal{V}_m$ we fix a point $y_V \in V$. We may assume that $y_V \ne y_U$ for $V \ne U$. Denote by

$$S_x = \sum_{U \in \mathcal{V}_m} \prod_{i=1}^{m} d(x, p^i(U) \setminus p^{i-1}(U)).$$

For every $x \in X$ we define a probability measure $a^n(x)$ supported in X as follows

$$a^n(x) = \sum_{V \in \mathcal{V}_m} \alpha_V^n(x) \delta_{y_V}$$

where δ_y is the Dirac measure supported by y and

$$\alpha_V^n(x) = \frac{\prod_{i=1}^{m} d(x, p^i(V) \setminus p^{i-1}(V))}{S_x}.$$

Let $\mathcal{V}_{x,y} = \{V \in \mathcal{V}_m \mid x \in V \text{ or } y \in V\}$. Note that $|\mathcal{V}_{x,y}| \leq 2^{m+1}$. Note that

$$\|a^n(x) - a^n(y)\|_1 = \sum_{V \in \mathcal{V}_m} |a_V^n(x) - a_V^n(y)| = \sum_{V \in \mathcal{V}_{x,y}} |a_V^n(x) - a_V^n(y)| \leq$$

$$\sum_{V \in \mathcal{V}_{x,y}} \left| \frac{\prod_{i=1}^m d(x, p^i(V) \setminus p^{i-1}(V)) - \prod_{i=1}^m d(y, p^i(V) \setminus p^{i-1}(V))}{S_x} \right| +$$

$$+ \sum_{V \in \mathcal{V}_{x,y}} \left| \frac{\prod_{i=1}^m d(y, p^i(V) \setminus p^{i-1}(V))}{S_x} - \frac{\prod_{i=1}^m d(y, p^i(V) \setminus p^{i-1}(V))}{S_y} \right|$$

$$\leq \sum_{V \in \mathcal{V}_{x,y}} \frac{d(x,y)^m}{S_x} + \sum_{V \in \mathcal{V}_{x,y}} \frac{\prod_{i=1}^m d(y, p^i(V) \setminus p^{i-1}(V))}{S_y} \frac{|S_x - S_y|}{S_x} \leq$$

$$\frac{2^{m+1}}{S_x} d(x,y)^m + \frac{2^{m+1}}{S_x} d(x,y)^m \leq \frac{2^{m+2}}{R_m} d(x,y)^m \leq \frac{1}{n} d(x,y)^m.$$

This implies that $\|a^n(x) - a^n(y)\|_1 \leq 1/n$ for all n provided $d(x,y) \leq 1$.

Now we assume that X is geodesic. Then $\|a^n(x) - a^n(y)\|_1 \leq K/n$ for all n, if $d(x,y) \leq K$. Thus, condition (2) of the property A is satisfied.

The measures $a^n(x)$ have uniformly bounded supports since the cover $\mathcal{V}_m$ is uniformly bounded.

If X is not geodesic, we assume, without loss of generality, that X is discrete. Then we use Proposition 3.3 and isometrically embed X into a geodesic metric space X'. Since, by what is proved above, X' has property A, the space X has also property A, by [Ro]. $\qquad \square$

3.5. **Theorem** (Finite Sum Theorem). *Let Z be a metric space such that $Z = X \cup Y$, where X and Y satisfy sFDC. Then Z also satisfies sFDC.*

Proof. Let $R_1 < R_2 < R_3 < \ldots$. We consider an R_1-decomposition $Z = (\cup \mathcal{Y}_1) \cup (\cup \mathcal{Y}_2)$, where $\mathcal{Y}_1 = \{X\} \cup \{\{z \in Z \mid (2k-1)(R_1+1) < d(z,X) \leq 2k(R_1+1)\} \mid k \in \mathbb{N}\}$, $\mathcal{Y}_2 = \{\{z \in Z \mid 2k(R_1+1) < d(z,X) \leq (2k+1)(R_1+1)\} \mid k \in \mathbb{N}\}$. Then the assertion follows from the fact that X and every subspace of Y satisfy sFDC. $\qquad \square$

By a slight modification, one can define the notion of sFDC for the metric families. We say that a metric family $\mathcal{X}$ satisfies the sFDC, if, for any $R_1 < R_2 < \ldots$, there exist $n \in \mathbb{N}$ and metric families $\mathcal{X}_i$, $i = 1, \ldots, n$, such that $\mathcal{X}_1 = \mathcal{X}$, $\mathcal{X}_n$ is a bounded family and $\mathcal{X}_{i+1}$ is r_i-decomposable over $\mathcal{X}_i$ for all $i = 1, \ldots, n-1$. Clearly, a family $\{X\}$

has the sFDC if and only if so does the space X. Also, any subfamily of a family that has the sFDC, has the sFDC as well.

3.6. **Theorem** (Sum Theorem). *Let X be a metric space such that $X = \cup \mathcal{X}$, where $\mathcal{X}$ has sFDC and the following condition holds: for any $r > 0$, there exists $Y(r) \subset X$ such that $Y(r)$ has sFDC and the family $\mathcal{X}(r) = \{X \setminus Y(r) \mid X \in \mathcal{X}\}$ is r-disjoint. Then X also satisfies sFDC.*

Proof. Given $R_1 < R_2 < \ldots$, find $Y(R_1) \subset X$ such that $Y(R_1)$ has sFDC and the family $\mathcal{X}(R_1)$ is R_1-disjoint. We see that X is R_1-decomposable over the family $\{Y(R_1)\} \cup \mathcal{X}(R_1)$ and $X = \cup \{Y(R_1)\} \cup (\cup \mathcal{X}(R_1))$.

Since the space $Y(R_1)$ and the family $\mathcal{X}$ have the sFDC, we conclude that there exist a natural number n and families $\mathcal{X}_i$, $i = 2, \ldots, n$, such that $\{Y(R_1)\} \cup \mathcal{X}(R_1)$ is R_2-decomposable over $\mathcal{X}_2$, $\mathcal{X}_i$ is R_{i+1}-decomposable over $\mathcal{X}_{i+1}$ for all $i = 2, \ldots, n - 1$, and $\mathcal{X}_n$ is a bounded family. $\square$

3.7. **Example.** In view of Proposition 2.3 the examples of groups with sFDC come from examples of groups with FDC like $\oplus_{i=1}^{\infty} \mathbb{Z}$ with the metric

$$d((x_i), (y_i)) = \sum_{i=1}^{\infty} i|x_i - y_i|$$

(see [NY]). The natural groups to investigate here are $\mathbb{Z} \wr \mathbb{Z}$ with the word metric and $\oplus_{i=1}^{\infty} \mathbb{Z}$ with the metric

$$d((x_i), (y_i)) = \sum_{i \in I} |x_i - y_i| + i$$

where $I = \{i \in \mathbb{N} \mid x_i \neq y_i\}$.

In view of Theorem 3.4 the examples of groups without sFDC come from groups without property A. Thus, Gromov monster groups are such [Gr2],[AD]. The Thompson group F with the word metric with respect to the following infinite generating set [WCh]

$$F = \langle x_0, x_1, x_2, \cdots \mid x_n x_k = x_k x_{n+1} \text{ for all } k < n \rangle.$$

4. GAME THEORETIC ASYMPTOTIC PROPERTY C

In the spirit of the FDC, one can define a game theoretic version of property C.

4.1. **Definition.** We say that a metric space X has the game theoretic asymptotic property C if there is a winning strategy for player I in the following game. Player II challenges player I by choosing $R_1 > 0$, then

player I chooses an R_1-disjoint uniformly bounded family $\mathcal{U}_1$. Then player II chooses $R_2 > 0$ and player I chooses an R_1-disjoint uniformly bounded family $\mathcal{U}_2$ and so on. Player I wins if there is k such that the family $\cup_{i=1}^{k}\mathcal{U}_i$ is a cover of X.

Let $\mathcal{V}$ be a family of nonempty subsets of a metric space X and $R > 0$. We say that $V, W \in \mathcal{V}$ are R-connected if there exist $V = V_0, V_1, \ldots, V_n = W$ in $\mathcal{V}$ and $y_0 \in V_0$, $x_i, y_i \in V_i$, $i = 1, \ldots, n-1$, and $x_n \in V_n$ such that $d(y_i, x_{i+1}) \le R$ for every $i = 0, \ldots, n-1$. We denote by $\mathcal{V}^R$ the family whose elements are the unions of the equivalence classes of the R-connectedness relation.

4.2. Proposition. *A space X has the game theoretic asymptotic property C if and only if* asdim $X = 0$.

Proof. Suppose that asdim $X = 0$. Then given $R > 0$, player I is able to find a uniformly bounded R-disjoint cover of X.

Now, suppose that asdim $X > 0$. There is $R > 0$ for which there is no R-disjoint uniformly bounded cover of X. Let $R_1 = R+1$. Suppose that player I has made n moves and chooses R_i-disjoint uniformly bounded families $\mathcal{U}_i$, $i = 1, \ldots, n$. Then player II takes $R_{n+1} = R_n + \text{mesh}(\mathcal{U}_n)$.

Suppose that there is k such that $\mathcal{U} = \cup_{i=1}^{k}\mathcal{U}_i$ is a cover of X.

For every $A \in \mathcal{U}_k$, let $A' = \cup\{B \in \mathcal{U}_{k-1} \mid d(A, B) < R_k/4\}$. Define

$$\mathcal{V}_{k-1} = \{A' \mid A \in \mathcal{U}_k\} \cup \{B \in \mathcal{U}_{k-1} \mid B \cap A' = \emptyset \text{ for all } A \in \mathcal{U}_k\}.$$

From the choice of R_k it follows that the family $\mathcal{V}_{k-1}$ is R_{k-1}-disjoint, uniformly bounded and $\cup\mathcal{V}_{k-1} = \cup(\mathcal{U}_k \cup \mathcal{U}_{k-1})$.

Proceeding as above one can construct families $\mathcal{V}_i$, $i = k - 1, k - 2, \ldots, 2, 1$, with the following properties:

(1) $\mathcal{V}_i$ is uniformly bounded;
(2) $\mathcal{V}_i$ is R_i-disjoint;
(3) $\cup\mathcal{V}_i = \cup(\mathcal{U}_k \cup \mathcal{U}_{k-1} \cup \cdots \cup \mathcal{U}_i)$.

Therefore, the family $(\cup_{i=1}^{k}\mathcal{U}_i)^{R_1}$ is an R-disjoint uniformly bounded cover of X and we obtain a contradiction.

$\square$

4.3. Question. Does the group $G = \bigoplus_{i=1}^{\infty}\mathbb{Z}$ with the proper metric

$$d((x_i), (y_i)) = \sum_{i=1}^{\infty} i|x_i - y_i|$$

have asymptotic property C?

We belive that the answer to this question is negative. It was proven in [NY], Proposition 2.9.1 that G has FDC.

5. Game theoretic approach in classical dimension theory

We recall that a space X is called countable dimensional if it can be presented as the countable union of 0-dimensional subsets.

5.1. **Lemma.** *Suppose that for a compactum X for every $\epsilon > 0$ there is a disjoint family of open sets $U_1, \ldots U_k$ of $diameter(U_i) < \epsilon$ with $X \setminus W_\epsilon$ countable dimensional where $W_\epsilon = \cup_{i=1}^k U_k$. Then X is countable dimensional.*

Proof. Let $X_n = X \setminus W_{1/n}$. Note that
$$X = \cup_{n=1}^\infty W_{\frac{1}{n}} \cup (\cap_{n=1}^\infty X_n).$$
We show that $F = \cap_{n=1}^\infty X_n$ is 0-dimensional. For that we show that F is homeomorphic to a subset of 0-dimensional compactum obtained as the inverse limit of a sequence
$$Y_1 \overset{\phi_1^2}{\leftarrow} Y_2 \overset{\phi_2^3}{\leftarrow} Y_3 \leftarrow \ldots.$$
Here Y_1 is the disjoint union $\coprod \overline{U_i^1}$ of the closures in X of elements of the ϵ-family with $\epsilon = 1$, Y_2 consists of the disjoint union of the closures of intersections $U_i^1 \cap U_j^2$ with the bonding map $\phi_1^2 : Y_2 \to Y_1$ the union of the inclusions and so on. □

The following notion is introduced by Haver [H].

A metric space (X, d) is said to have *property C* if for each sequence of positive numbers $\{\varepsilon_i\}_{i=1}^\infty$, there exists a sequence of disjoint collections of open sets $\{\mathcal{U}^i\}_{i=1}^\infty$ such that $\text{mesh}\,\mathcal{U}^i < \varepsilon_i$, $i \in \mathbb{N}$, and $\cup_{i=1}^\infty \mathcal{U}^i$ is a cover of X.

5.2. **Definition.** We say that a metric space X has the *game theoretic property C* if there is a winning strategy for player I in the following game. Player II challenges player I by choosing $\varepsilon_1 > 0$, then player I chooses a disjoint family of open sets $\mathcal{U}_1$ with $\text{mesh}\,\mathcal{U}_1 < \varepsilon_1$. Then player II chooses $\varepsilon_1 > 0$ and player I chooses a disjoint family of open sets $\mathcal{U}_2$ with $\text{mesh}\,\mathcal{U}_2 < \varepsilon_2$.and so on. Player I wins if the family $\cup_{i=1}^k \mathcal{U}_i$ is a cover of X.

5.3. **Theorem.** *A compact metric space X has the game theoretic property C if and only if it is countable dimensional.*

Proof. First, we show that every compact countable dimensional space X has the game theoretic property C. Let $X = \cup_{i=1}^\infty Y_i$, where all Y_i are zero-dimensional.

Let $\varepsilon_1 > 0$. Find a disjoint cover $\mathcal{V}_1$ with $\text{mesh}\,\mathcal{V}_1 < \varepsilon_1/2$ of Y_1 by clopen (in Y_1) subsets. Remark that, for every $V \in \mathcal{V}_1$ and every

$x \in V$, we have $d(x, (\cup \mathcal{V}_1) \setminus V) > 0$. We let

$$U_V = \cup \{ B_{\min\{\varepsilon_1/4, d(x, (\cup \mathcal{V}_1) \setminus V)\}}(x) \mid x \in V \}.$$

The family $\mathcal{U}_1 = \{U_V \mid V \in \mathcal{V}_1\}$ is a disjoint family of open in X sets such that mesh $\mathcal{U}_1 < \varepsilon_1$ and $\cup \mathcal{U}_1 \supset Y_1$.

Given $\varepsilon_i > 0$, one can similarly find a disjoint family $\mathcal{U}_i$ of open in X sets such that mesh $\mathcal{U}_i < \varepsilon_i$ and $\cup \mathcal{U}_i \supset Y_i$. Then the family $\cup_{i=1}^{\infty} \mathcal{U}_i$ is a cover of $\cup_{i=1}^{\infty} Y_i = X$. Since X is compact, there exists $k \in \mathbb{N}$ such that $\cup_{i=1}^{k} \mathcal{U}_i$ is a cover of X.

Now, assume that X is not countable dimensional. By Lemma 5.1 there is ϵ_1 such that for every open disjoint ϵ_1 family $U_1, \ldots U_k$ the complement is not countable dimensional. Then the the first player (Bad) pick up this ϵ_1. No matter what first player (Good) does the complement will not be countable dimensional. Therefore by Lemma 5.1 there is number ϵ_2 and so on. The process will never stop, i.e, the Good player will never win.

$\square$

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